

Localization in Fock space: A finite size scaling hypothesis for many particle excitation statistics

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Abstract

The concept of localization in Fock space is extended to the study of the many particle excitation statistics of interacting electrons in a two dimensional quantum dot. In addition, a finite size scaling hypothesis for Fock space localization, in which the excitation energy replaces the system size, is developed and tested by analyzing the spectral properties of the quantum dot. This scaling hypothesis, modeled after the usual Anderson transition scaling, fits the numerical data obtained for the interacting states in the dot. It therefore attests to the relevance of the Fock space localization scenario to the description of many particle excitation properties.

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The concept of localization in Fock space has been recently utilized [1] to explain the transition in the width of excited states measured in tunneling conductance experiments for quantum dots [2]. Due to electron-electron interactions the quasi-particle states are coupled, which leads to their finite width. At a critical value of interaction strength and excitation energy the width becomes essentially infinite and the quasi-particle states can no longer be resolved.

In this letter we show that Localization in Fock space explains also the recently reported [3–6] transition in the level statistics of excited many particle states as a function of the interaction strength. For small values of the interaction the many-particle states are localized in the non-interacting Fock space, namely each interacting state is composed out of a small number of non-interacting eigenvectors. Above some critical value they are extended, i.e., composed out of a large number of non-interacting eigenstates. This leads to a transition in the statistical properties of the energies from Poisson to Wigner statistics, which as in Anderson localization is a true second order phase transition. We will demonstrate this by constructing a finite-energy scaling theory which plays the role of finite size scaling familiar in standard localization theory.

A typical gray-scale map of the statistical properties in the space of interaction strength and excitation energy is presented in Fig. 1. at low excitation energies the statistics remains Wigner over a large range of interaction strength. This will be discussed in detail elsewhere. A transition from Poisson to Wigner and back to Poisson as the interaction strength is increased is evident for intermediate excitation energies. The second (Wigner-Poisson) transition is the result of Wigner crystallization [4]. In this letter we will show that the first transition has the characteristics of a localization transition in Fock space.

A localization transition is characterized by a single parameter scaling behavior [7], which results in only three possible types of statistics in the thermodynamic limit: Poisson in the localized regime, Wigner in the extended one and a possible third type of statistics at the transition point [8,9]. As shown by Shklovskii et. al. [8], for a finite system the transition between the two types of statistics is gradual and may be characterized by a finite size

scaling behavior, which can be used to locate the critical point and the critical exponent.

For the Anderson localization transition the scaling parameter is the ratio of the disorder dependent localization length to the linear dimension of the system. In the Fock space localization transition we show that the relevant scaling variable is the ratio of the localization length (which depends on the interaction strength and the two-particle density) to the linear “dimension” of the system in Fock space, which depends mainly on the excitation energy. The latter is surprising, since one would expect the linear dimension of the system to depend on the number of particles and the number of sites. As has been demonstrated in Ref. [5], no clear finite size scaling behavior is observed as function of the number of sites or particles. This is the result of the unusual “geometry” of the interaction coupling which leads to a Cayley tree structure in Fock space where the number of generations depends on the excitation energy [1].

Let us begin by defining the Fock space. The many-electron second quantization Hamiltonian:

$$H = H_0 + H_{int} = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{i,j,k,l} U_{i,j,k,l} c_j^\dagger c_l^\dagger c_k c_i, \quad (1)$$

where c_i^\dagger is the creation operator of particle in the i -th single electron state, ϵ_i is its energy, and $U_{i,j,k,l}$ is the interaction matrix element, has a set of eigenvectors $|\Psi_N^j\rangle$ and eigenvalues E_N^j , where N is the number of electrons and j is an index ordering the states by ascending eigenvalues. There are $M = \binom{S}{N}$ many-electron states, where S is the number of single electron states.

For the non-interacting ($H_{int} = 0$) case the ground-state eigenvector and eigenvalue are

$$\begin{aligned} |\Phi_N^0\rangle &= c_N^\dagger \dots c_1^\dagger |0\rangle \\ E_N^0 &= \sum_{i=1}^N \epsilon_i, \end{aligned} \quad (2)$$

here $|0\rangle$ is the vacuum state. An excitation composed of m electron-holes pairs may be represented as:

$$|\Phi_N^{\alpha_1 \dots \alpha_{2m}}\rangle = c_{\alpha_{2m}}^\dagger \dots c_{\alpha_{m+1}}^\dagger c_{\alpha_m} \dots c_{\alpha_1} |\Psi_N^0\rangle$$

$$E_N^{\alpha_1 \dots \alpha_{2m}} = E_N^0 - \sum_{i=1}^m \epsilon_{\alpha_i} + \sum_{i=m+1}^{2m} \epsilon_{\alpha_i}. \quad (3)$$

These excitations may also be numbered in ascending order $|\Phi_N^j\rangle$ according to their energy.

For the interacting ($H_{int} \neq 0$) case one may write the eigenvectors $|\Psi_N^j\rangle$ and their excitation energy $E_N^j - E_N^0$ (again arranged in ascending order) in the following way:

$$|\Psi_N^j\rangle = \sum_{m=1}^N \sum_{\alpha_1 \dots \alpha_{2m}} A^{\alpha_1 \dots \alpha_{2m}} |\Phi_N^{\alpha_1 \dots \alpha_{2m}}\rangle = \sum_{k=1}^M A_k |\Phi_N^k\rangle, \\ \varepsilon_j = E_N^j - E_N^0. \quad (4)$$

An interesting question is how are these interacting eigenstates composed out of the non-interacting Fock space eigenvectors. It is natural to expect that for small values of interaction and excitation energy an interacting eigenstate $|\Psi_N^j\rangle$ will be composed of a small number of non-interacting eigenvectors $|\Phi_N^k\rangle$ where k is in the vicinity of j , while for large values of the interaction $|\Psi_N^j\rangle$ will be composed of many non-interacting eigenvectors. Altshuler et. al. [1] have termed this transition “localization in Fock space” and predicted that it should exhibit the characteristics of an Anderson transition on a Cayley tree.

We will use two measures for the Anderson transition: (i) the energy level statistics, and (ii) the inverse participation ratio in Fock space. A convenient way to characterize the change in the statistics of a system proposed by Shklovskii et. al. [8] is to study the parameter γ defined as

$$\gamma = \frac{\int_2^\infty P(s)ds - e^{-\pi}}{e^{-2} - e^{-\pi}}, \quad (5)$$

where $P(s)$ is the distribution of the normalized level spacings $s = E_N^j - E_N^{j-1} / \langle E_N^j - E_N^{j-1} \rangle$, where $\langle \dots \rangle$ denotes an average over different realizations of disorder. For an infinite system γ changes sharply from $\gamma = 1$ in the localized regime to $\gamma = 0$ in the extended regime. For a finite system the change is gradual. One can then use a finite size scaling hypothesis to identify the transition point and its critical indices [8]. The inverse participation ratio P , defined as

$$P = \sum_{k=1}^M |A_k|^4, \quad (6)$$

is expected to change in the Anderson transition for an infinite system from P which is a function of the localization length in the localized regime to $P = 0$ in the extended regime.

The argument for a localization transition as one increases the interaction strength follows the usual Anderson transition argument [10]. The strength of the hopping term V coupling neighboring sites is compared with the inverse of the density of states of the neighboring site, $K\nu$, where $\nu = 1/W$, W being the strength of disorder, and K is the connectivity. A localization transition will occur at a critical value

$$Z_c = K\nu V = \frac{KV}{W} \sim 1. \quad (7)$$

As noted by Altshuler et. al. [1], a similar argument may be applied to Fock space localization. The two-body interaction couples non-interacting eigenvectors which are different by up to 2 electron-hole pairs. Assuming that the main coupling is between states differing by a single electron-hole pair [1], the average strength of coupling

$$\tilde{U} = \langle\langle \Phi_N^{\alpha'_1 \dots \alpha'_{2m}} | H_{int} | \Phi_N^{\alpha_1 \dots \alpha_{2m}} \rangle\rangle_{\alpha'_1, \alpha'_{2m}}, \quad (8)$$

where $\langle\dots\rangle_{\alpha'_1, \alpha'_{2m}}$ denotes averaging over all possible electron-hole pairs. The density of such coupled states at excitation energy ε is denoted by $K\nu_2(\varepsilon)$. Thus

$$Z_c \sim K\nu_2(\varepsilon)\tilde{U}. \quad (9)$$

This argument is similar to the one used by Imry in the context of two particle state delocalization [11,6]. A more careful consideration of the geometry of the connected states reveals that it is similar to a Cayley tree for which K should be replaced by $K \ln K$ [12]. Since the effective connectivity $K \sim g$ where g is the dimensionless conductance, this is a correction of order of $\ln g$. Altshuler et. al. [1] concluded that due to the Cayley tree structure, an intermediate region $Z_c/K > \nu_2(\varepsilon)\tilde{U} > Z_c/K \ln K$ between the localized region $\nu_2(\varepsilon)\tilde{U} < Z_c/K \ln K$ and the extended region $Z_c/K < \nu_2(\varepsilon)\tilde{U}$ should exist, which is non-ergodic. Using a non-linear sigma model for the Cayley tree Mirlin and Fyodorov found no intermediate region [13].

Here we would like to extend this Fock space localization picture to include one of the most useful concepts in the Anderson localization picture, namely that of finite size scaling. In the usual Anderson picture $\gamma(KV/W, L) = f[L/\xi(KV/W)]$ [8] where L is the sample size, $\xi(KV/W)$ is the localization length and f is a scaling function. Near the critical value it is expected to behave as $\gamma(KV/W, L) = \gamma(Z_c, L) + C[Z_c - (KV/W)]L^{1/\delta}$, where C is a constant and δ is the critical exponent (usually denoted by ν). From the above discussion it is natural to assume that $K\nu_2(\varepsilon)\tilde{U}$ will replace KV/W for the Fock space localization. The size of the system, i.e., the number of generations in the Cayley tree, is equivalent to the number of electron-hole pairs which can be generated at a given excitation energy [1]. Thus $L \sim n_{max} \sim \sqrt{\varepsilon/\Delta}$, where Δ is the single electron level spacing. Hence we expect

$$\gamma(K\nu_2(\varepsilon)\tilde{U}, \varepsilon) \sim \gamma(Z_c, \varepsilon) + C(Z_c - K\nu_2(\varepsilon)\tilde{U})\varepsilon^{1/2\delta}. \quad (10)$$

For $P(K\nu_2(\varepsilon)\tilde{U}, \varepsilon)$, the situation is different. According to the Breit-Wigner formula [14,15] in the localized regime the inverse participation ratio depends on U and $K\nu_2(\varepsilon)$ but not on the linear dimension ε , as long as $\xi(K\nu_2(\varepsilon)\tilde{U}) < L$. Once $\xi(K\nu_2(\varepsilon)\tilde{U}) > L$, P will also depend on ε . Since $\xi \propto (Z - Z_c)^{-\delta}$ we expect that P for different values of ε to coalesce for $Z \ll Z_c$ and to begin to fan out for $Z \sim Z_c$.

In order to check this finite size scaling hypothesis we have numerically calculated γ and P for a specific many particle tight-binding Hamiltonian:

$$H = \sum_{k,j} \omega_{k,j} a_{k,j}^\dagger a_{k,j} - V \sum_{k,j} (a_{k,j+1}^\dagger a_{k,j} + a_{k+1,j}^\dagger a_{k,j}) + h.c + H_{int}, \quad (11)$$

where $\omega_{k,j}$ is the energy of a site (k, j) , chosen randomly between $-W/2$ and $W/2$ with uniform probability, and V is a constant hopping matrix element. The interaction Hamiltonian is given by:

$$H_{int} = U \sum_{k,j>l,p} \frac{a_{k,j}^\dagger a_{k,j} a_{l,p}^\dagger a_{l,p}}{|\vec{r}_{k,j} - \vec{r}_{l,p}|/b} \quad (12)$$

where $U = e^2/b$ and b is the lattice unit.

We consider a 4×3 dot with $S = 12$ sites and $N = 4$ electrons. The $M \times M$ Hamiltonian matrix is numerically diagonalized and all the eigenvectors $|\Psi_N^j\rangle$ and eigenvalues E_N^j are

obtained. The strength U of the interaction is varied between $0 - 30V$. The disorder strength is chosen to be $W = 5V$ in order to assure metallic behavior for the non-interacting case. This is important in order to obtain a reasonable connectivity $K \sim g$. For each value of U , the results are averaged over 1000 different realizations of disorder. $\gamma(U, \varepsilon)$ is directly calculated from the level spacing distribution for different values of U and ε , while $P(U, \varepsilon)$ is calculated from the overlap of $|\Psi_N^j\rangle$ with the non-interacting eigenvectors.

In Fig. 2 the average value of $|A_{k-i}|^2$ for an excited state $|\Psi_N^i\rangle$ (where i is chosen so that $\varepsilon_i \sim 8V$ in the non-interacting case) as a function of the interaction strength is plotted. As expected for small values of interaction $|A_{k-i}|^2$ is strongly peaked around $k = i$. Thus, the interacting state is composed of a small number of non-interacting states of similar energies, which has on the average a Breit-Wigner form [14,15]. As the interaction increases the interacting state is composed of an increasing number of non-interacting states although there is no obvious sharp transition. Similar features appear in $|A_{k-i}|^2$ for a single realization, although the curves are noisier.

In order to locate the transition point we use finite size scaling. First we calculate $K\nu_2(\varepsilon) \sim J(\varepsilon)$ by evaluating the number $J(\varepsilon)$ of non-interacting states in a given energy region which are coupled to each other by a single electron-hole transition. The numerically calculate $J(\varepsilon)$ is plotted in the inset in Fig. 2. For small values of ε , J increases linearly, while near the middle of the excitation band it flattens out. Since the interaction strength $\tilde{U} \propto U$ one may replace the scaling parameter $K\nu_2(\varepsilon)\tilde{U}$ by $J(\varepsilon)U$.

The parameter $\gamma(J(\varepsilon)U, \varepsilon)$ is shown in Fig. 3 for values of $5V < \varepsilon < 10V$. This range was chosen in order to avoid the special region of low excitations and to avoid the symmetric region above the middle of the band (see Fig. 1). Indications of finite size scaling behavior are observed. For large values of $J(\varepsilon)U$, γ becomes smaller as the size ε increases. On the other hand for small values of $J(\varepsilon)U$, γ tends to be larger as the size increases. The point at which the lines intersect is the transition point. The quantity $\gamma(J(\varepsilon)U, \varepsilon)$ is then considered as a certain scaling function $f(\zeta)$ of the scaling variable $\zeta = \varepsilon^{1/2\delta}(Z_c - J(\varepsilon)U)$.

Practically, it is useful to shift the variable ζ to $y(\zeta) = \frac{\zeta-a-b}{b-a}$ where a and b are respectively the minimum and maximum values assumed by ζ . Evidently, $y(\zeta)$ ranges between -1 and 1 . Then one expands $f(\zeta)$ in a series of Chebyshev polynomials $T_n[y(\zeta)]$ ($n = 0, 1, 2, \dots, Q$). Minimization of the set of differences $|f(\zeta) - \gamma(J(\varepsilon)U, \varepsilon)|$ results in the unknowns Z_c , δ and the expansion coefficients (namely, the scaling function itself). In all cases, it is sufficient to cut off the number of polynomials at $Q = 10$. The best fit is obtained for $Z_c = 1.3 \pm 0.4$ and $\delta = 1 \pm 0.3$. The function $f(\zeta)$ and the original data points are plotted in the inset of Fig. 3.

The results for the inverse participation ratio P are presented in Fig. 4. For $J(\varepsilon)U < Z_c$ all lines coalesce, since no dependence on the linear dimension ε is expected. For $J(\varepsilon)U \sim Z_c$ the dependence on ε is apparent and the lines corresponding to the different values of ε fan out. The region in which the lines fan out correspond to $Z_c \sim 1.5$, which is close to the values obtained for γ . It is interesting to note that $\gamma(Z_c) \sim 0.6$ is considerably higher than for the Anderson transition [8].

In this analysis we have not clearly identified a crossover region between the localized and extended regions predicted in Ref. [1]. This may be explained by the results of Ref. [13], although we do not see the jump in P predicted there. We speculate that the additional couplings between generations which are not taken into account in the pure Cayley tree picture tend to restore ergodicity, which is a key ingredient in the scaling picture. Only studies on much larger systems will answer this question.

In summary, Fock space localization has been shown to describe the features of the excitation statistics of many particle systems. We have proposed a finite size scaling hypothesis in which the role of system size in the usual Anderson scaling picture is played here by the excitation energy. The scaling hypothesis was tested for a specific tight-binding many particle Hamiltonian and resulted in satisfactory agreement.

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FIGURES

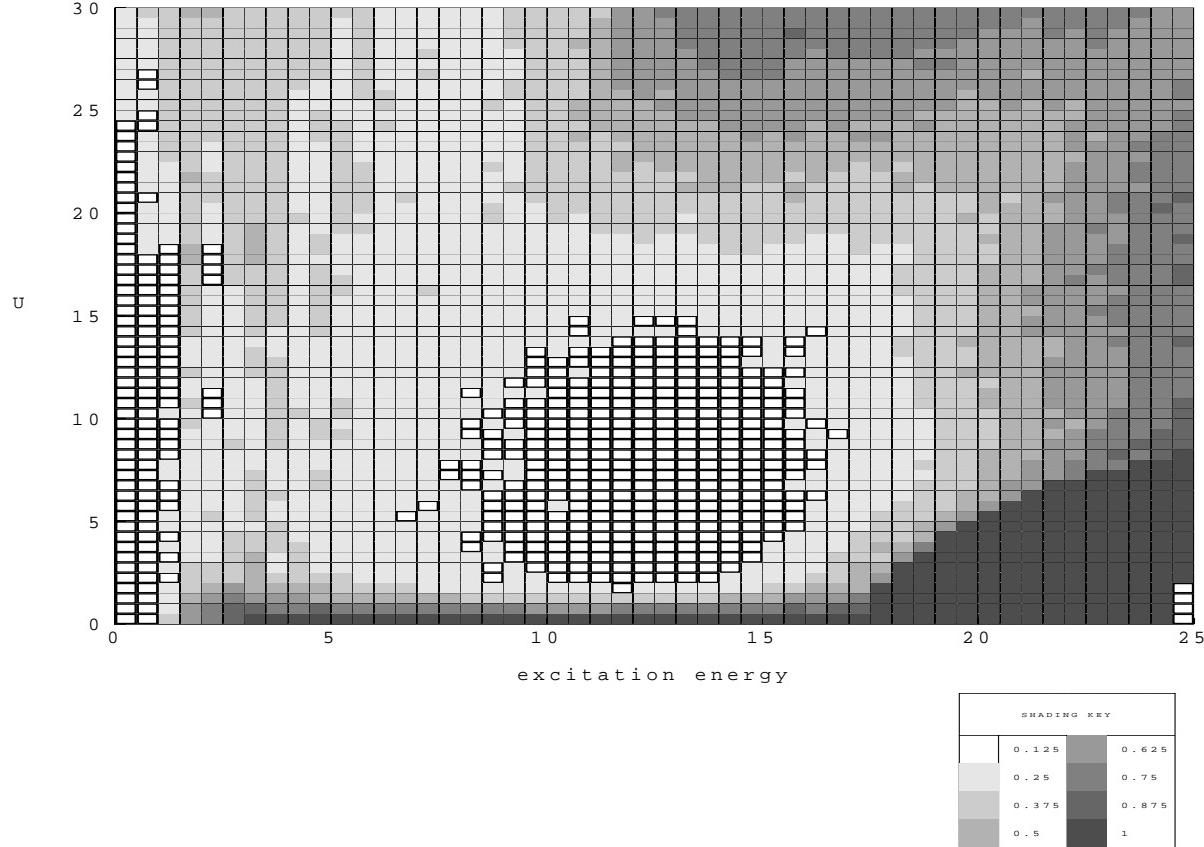


FIG. 1. A gray-scale map of the parameter γ defined in Eq. (5) for the many particle Hamiltonian given in Eq. (11). Low values of γ correspond to Wigner statistics, while values close to one correspond to Poisson statistics

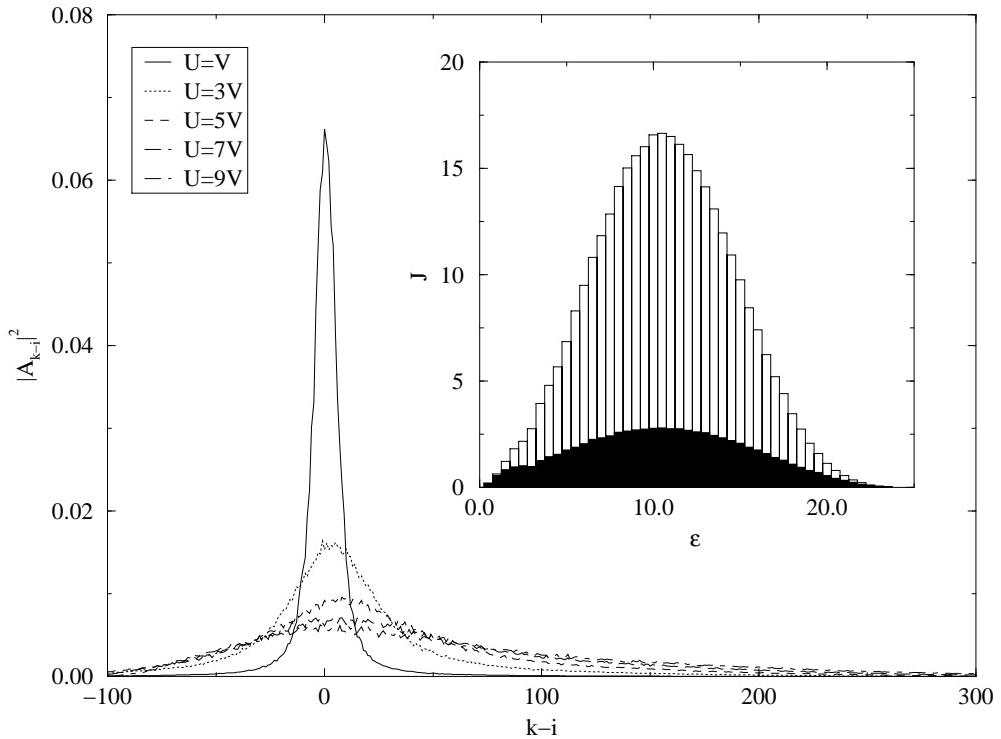


FIG. 2. The non-interacting Fock space composition of an interacting eigenvalue of excitation energy $\varepsilon \sim 8V$. The composition $|A_{k-i}|^2$ was averaged over 100 realizations of disorder. Inset: the number of states coupled by a single pair transition J within an energy bin (dark histogram) compared with the total number of states (white histogram).

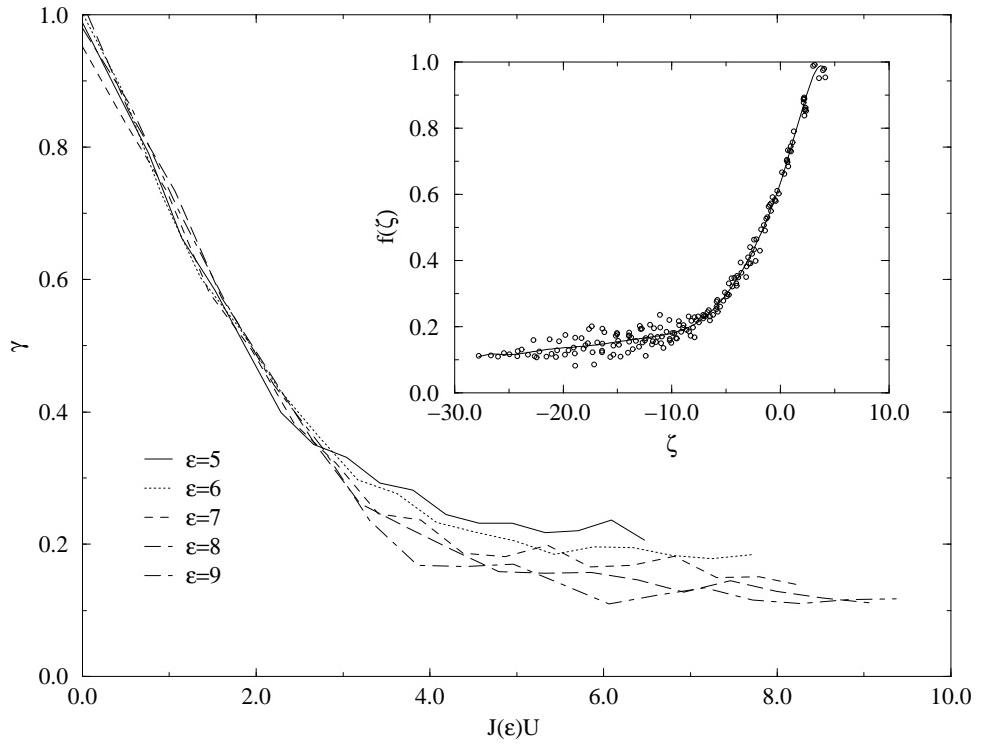


FIG. 3. The parameter γ for different values of interaction U and excitation energy ε . Inset: the scaling function $f(\zeta)$.

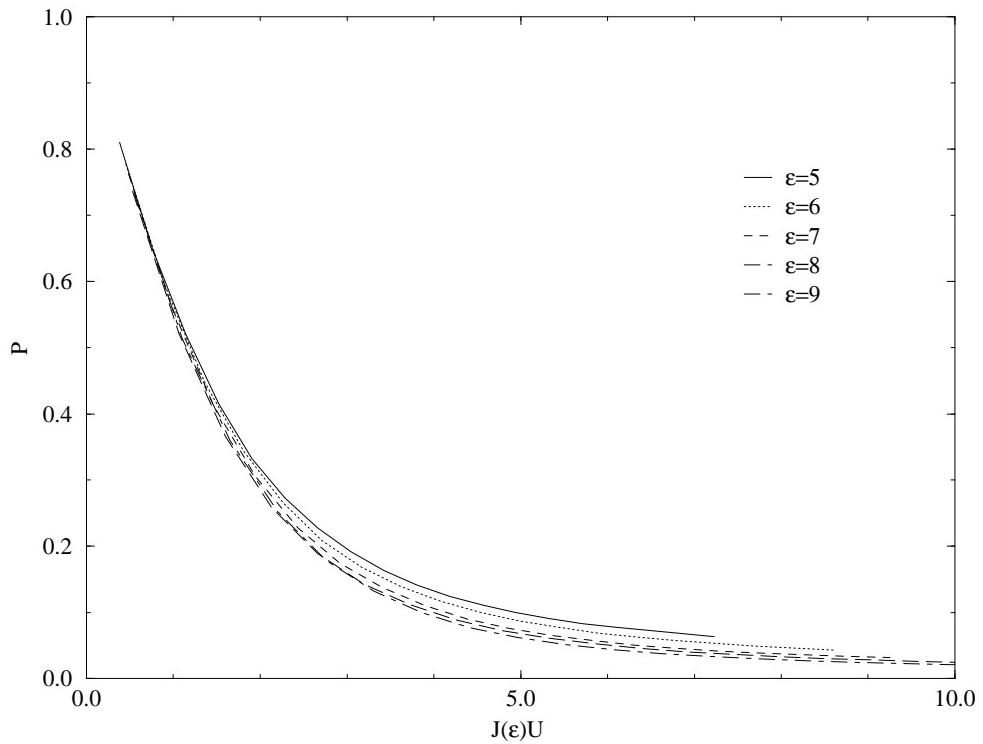


FIG. 4. The inverse participation ratio P or different values of interaction U and excitation energy ε .